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Optimal Noise Rejection in Structural Analysis by Means of Generalized Sampled-Data Hold Functions*

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Abstract. In this paper, a technique for optimal noise rejection, based on generalized sampled-data hold functions is applied to the control of civil engineering structures. The technique consists in suitably modulating the sampled outputs of the system under control by periodically varying functions in order to attenuate the effect of the disturbances on the system states to an acceptable level, by minimizing a quadratic cost function. This minimization is performed by feeding back the outputs of the system, which are assumed to be corrupted by measurement noise. Moreover, in the present paper, the robustness properties of the GSHF based optimal regulator is analyzed and guaranteed stability margins, expressed in terms of elementary cost and system matrices, are proposed for such a type of optimal regulators. The effectiveness of the method is demonstrated by various simulation results. The results of the paper can be used to assess the detrimental effect of noise on the closed-loop system and the tradeoff involved in assuring good sampled-data performance and sufficient robustness.

Key words: LQG control; Noise rejection; Robustness; Sampled-data control; Structural control

1. Introduction

Active control of structures, after its successful application in numerous aeronautical structures, where its potential as a versatile tool for the design of dynamically loaded flexible structures has been proved, has appeared as a new research field in Structural Mechanics. In this respect, the active control of civil engineering structures has received much attention in the past. Several well established control techniques, such as optimal control methods, disturbance rejection techniques, H^∞-control methods, etc., have been applied in order to control civil engineering structures (see [1–7] and references cited therein).

In recent years, many pieces of work treating design issues of linear systems by

* Dedicated to the memory of Professor P.D. Panagiotopoulos with our warmest prayers, may our Lord Jesus Christ rest his soul.

periodically time-varying and/or multirate sampled-data controllers have been reported in the literature [8–20]. The interest for such a type of control strategies is warranted by the new dimensions of flexibility of the design process offered by these control schemes, which also provide a series of remarkable advantages over ordinary time-invariant feedback strategies, such as state feedback, dynamic compensation and estimator based controllers (for an overview of these advantages see [12,16]). Among the most interesting control strategies of this type, is feedback control based on generalized sampled-data hold functions (GSHF). GSHF control has been proposed first in [12], and subsequently has successfully been applied in solving a variety of important control problems such as pole assignment [13], simultaneous controller design [14], exact model matching [15,17], adaptive control (decoupling, pole placement, model reference control) [16,19,20], intersample performance analysis [17], mixed H^2/H^4 control [18], etc.

In particular, in his inspired work [12], Kabamba proposed a GSHF based periodic controller, which suitably modulates the sampled outputs of the system under control by periodically varying functions, in order to solve, among other important control problems, the discrete optimal noise rejection problem for disturbed linear time-invariant continuous-time systems. Under certain conditions on the minimality of the system under control, the modulating functions can be tailored to a given system in such a way that for the sampled closed loop system the sensitivity of the state vector to disturbances and noise is minimized at the sampling instants. The solution of the problem proposed in [12] is structurally identical to that of the Kalman filter. However, the design of the optimal GSHF based regulator reduces to the solution of only one discrete algebraic Riccati equation, as compared to the separate solution of two Riccati equations which are necessary for designing an estimator based optimal regulator by Kalman filtering technique: one relative to the estimation of the state vector and the other relative to the computation of the optimal regulator. Obviously, this fact has beneficial influence on the computational complexity of the optimal noise rejection problem, since the method based on GSHF enjoys the efficacy of state feedback without the requirement of state estimation. On the other hand, in [12], the stability robustness of the GSHF based optimal regulator has been studied and some preliminary results regarding its stability margins have been presented. The results of [12], regarding stability robustness of multiloop GSHF based optimal regulators, consist of a certain lower bound for the minimal singular value of the closed-loop return difference matrix. However, this result is quite different from those reported in [21], for the case of discrete LQ regulators, since the proposed bound depends on the solution of the discrete algebraic Riccati equation associated with the GSHF based optimal noise rejection problem, and has not been expressed in terms of system and cost matrices as the bounds reported in [21].

In the present paper, the technique reported in [12] is applied in order to treat the optimal noise rejection problem in structural control. Our design objective here is to attenuate the detrimental effect of the disturbances (i.e., earthquake, impact of a ship vessel to a bridge pier, etc.) on the system states to an acceptable level, by minimizing a certain quadratic cost function. This minimization is performed by feeding back the outputs of the system, which are assumed to be corrupted by measurement noise. In particular, in the paper, we focus our attention to civil engineering structures and especially to the IPB 800 three-story, single bay, steel frame structure. Here, the structure studied, is considered to be assembled from finite elements obeying linear material laws and its displacements are small. After selecting a diagonal mass matrix, a continuous-time controllable and observable state space model of the structure is obtained. Note that controllability and observability of the continuous-time model of the structure are prerequisites for being able to apply the GSHF based approach to the structure under control. Several simulations of the proposed GSHF based technique for optimal noise rejection, are performed in the paper, for various values of the sampling period and of the covariance kernels of the disturbance and measurement noise. From these results it is verified that when the covariance kernel of the disturbance acting on the structure is small, the regulator gains are small, since the less the level of the disturbance acting on the system the smaller must be the control effort, for rejecting the disturbance. Furthermore, large regulator gains are expected when the sampling period is chosen to be small. On the other hand, small regulator gains are expected in cases where the covariance kernel of the noise is large. This reveals that the covariance of the sampled state vector will be large, indicating a poorly regulated system in this case. Finally, in the present paper, a more thorough analysis of the robustness properties of the GSHF based optimal regulator is presented. In particular, the behavior of the regulator's return difference matrix is investigated, and new, relatively simple, lower bounds for its minimal singular value, which are independent of the solution of the Riccati equation, are proposed, along the lines reported in [21,22]. On the basis of these bounds, new guaranteed stability margins for GSHF based optimal regulators, as measures of its stability robustness, are established. The proposed guaranteed stability margins are obtained on the basis of a fundamental spectral factorization equality, called the GSHF Return Difference Equality, and are expressed explicitly in terms of the elementary cost and system matrices. It is worth noticing, at this point, that our investigation on guaranteed stability margins of the GSHF based optimal regulator, is focused on a broad variety of important particular cases, for the elementary system and cost matrices. The reason for such a type of investigation, is due to the fact that, in our case, it is very difficult (if not impossible) to obtain a universal lower bound for the minimal singular value of the regulator's return difference matrix, as in the case of the continuous-time LQ regulator [23]. This difficulty stems from the entanglement of the solution of the Riccati equation, in the left hand side of the GSHF Return Difference Equality. The above theoretical results on robustness analysis of GSHF based optimal regulators, have been applied, in the paper, in order to obtain guaranteed stability margins of GSHF controlled civil engineering structures. It is verified, through simulation, that when the covariance kernel of the noise disturbing the system decreases, the robustness of the GSHF based optimal regulator is ameliorated, since a smaller disturbance covariance kernel correspond to smaller levels of system disturbances. In these cases, the controller withstands to smaller system variations, thus leading to a more robust closed-loop system.

2. Optimal noise rejection using generalized sampled-data hold functions

Consider the controllable and observable linear state space system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \zeta(kT_0) = \mathbf{y}(kT_0) + \mathbf{v}(kT_0) \tag{2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$ and $\mathbf{w}(t) \in \mathbb{R}^n$ are the state, control, output and disturbance vectors, respectively, T_0 is the sampling period, $\zeta(kT_0) \in \mathbb{R}^p$ is a discrete measurement vector and $\mathbf{v}(kT_0) \in \mathbb{R}^p$ is a discrete measurement noise vector. In (2.1), all the matrices have appropriate dimensions. It is supposed that $\mathbf{w}(t)$ and $\mathbf{v}(kT_0)$ are stationary zero-mean white Gaussian process with covariance kernels

$$\begin{aligned}\mathbb{E}[\mathbf{w}(t)\mathbf{w}^T(\tau)] = \mathbf{R}_\mathbf{w}\delta(t-\tau), \mathrm{t}, \tau \in \mathbb{R} \quad \mathbb{E}[\mathbf{v}(kT_0)\mathbf{v}^T(\ell T_0)] \\ = \mathbf{R}_\mathbf{v}\delta(k-\ell), k, \ell \in \mathbb{N}, \ \mathbf{R} > \mathbf{0}\end{aligned}$$

where by abuse of notation, $\delta(\cdot)$ represents the Dirac function in both the discrete-time and continuous-time case. System (2.1), will be acted upon by controls of the form

$$\mathbf{u}(t) = \mathbf{F}(t)\zeta(kT_0), t \in [kT_0, (k+1)T_0), k \geqslant 0, \mathbf{F}(t+T_0) = \mathbf{F}(t), \text{for } t \in [0, T_0) \tag{2.2}$$

where $\mathbf{F}(t)$ is a T_0-periodic integrable and bounded matrix of appropriate dimension representing a hold function. Then, the closed loop system has the form

$$\mathbf{x}[(k+1)T_0] = \mathbf{\Phi}\mathbf{x}(kT_0) + \mathbf{F}_\phi\zeta(kT_0) + \omega(kT_0), \ \zeta(kT_0) = \mathbf{C}\mathbf{x}(kT_0) + \mathbf{v}(kT_0)$$

or equivalently,

$$\mathbf{x}[(k+1)T_0] = (\mathbf{\Phi} + \mathbf{F}_\phi\mathbf{C})\mathbf{x}(kT_0) + \mathbf{F}_\phi\mathbf{v}(kT_0) + \omega(kT_0)$$

where

$$\begin{aligned}&\mathbf{\Phi} = \exp(\mathbf{A}T_0), \quad \mathbf{F}_\phi = \int_0^{T_0}\exp[\mathbf{A}(T_0-\lambda)]\mathbf{B}\mathbf{F}(\lambda)d\lambda, \ \omega(kT_0) \\ &\quad = \int_{kT_0}^{(k+1)T_0}\exp\{\mathbf{A}[(k+1)T_0-\lambda]\}\mathbf{w}(\lambda)d\lambda\end{aligned} \tag{2.3}$$

Relation (2.3) implies that $\omega(kT_0)$ is a stationary zero-mean white Gaussian process with covariance kernel

$$\mathbb{E}[\omega(kT_0)\omega^T(\ell T_0)] = \mathbf{R}_\omega\delta(k-\ell),\ k,\ \ell \in \mathbb{N},\ \mathbf{R}_\omega = \int_0^{T_0} \exp[\mathbf{A}(T_0-\lambda)]\mathbf{R}_\mathbf{w} \exp[\mathbf{A}^T(T_0-\lambda)]d\lambda$$

The optimal noise rejection problem via GSHF control, treated in this paper, is as follows: Given a symmetric positive definite weighting matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and a sampling period T_0, find a GSHF based optimal regulator of the form (2.2), in order to minimize the following cost function

$$J = \lim_{k\to\infty} \mathbb{E}[\mathbf{x}^T(kT_0)\mathbf{Q}\mathbf{x}(kT_0)]$$

The solution of the above problem has been established in [12], and it can be expressed in terms of a discrete Riccati equation, which resembles the Riccati equations appearing in discrete Kalman filtering. More precisely, it has been proven in [12] that, if the triplet $(\mathbf{A},\mathbf{B},\mathbf{C})$ is minimal, then for almost all $T_0 > 0$, the optimal noise rejection problem is solvable. Its solution is

$$\mathbf{F}(t) = \mathbf{B}^T\exp[\mathbf{A}^T(T_0 - t)]\mathbf{W}^{-1}(\mathbf{A},\mathbf{B},\mathrm{T}_0)\tilde{\mathbf{F}}_\phi \tag{2.4}$$

where

$$\mathbf{W}(\mathbf{A},\mathbf{B},T_0) = \int_0^{T_0} \exp[\mathbf{A}(T_0-\lambda)]\mathbf{B}\mathbf{B}^T \exp[\mathbf{A}^T(T_0-\lambda)]d\lambda,$$

$$\tilde{\mathbf{F}}_\phi = -\mathbf{\Phi}\mathbf{K}\mathrm{C}^T(\mathbf{C}\mathbf{K}\mathbf{C}^T + \mathbf{R}_\mathbf{v})^{-1}$$

and $\mathbf{K} \in \mathbb{R}^{n\times n}$ is the unique positive definite solution of the discrete Riccati equation

$$\mathbf{\Phi}\mathbf{K}\mathbf{\Phi}^T - \mathbf{\Phi}\mathbf{K}\mathbf{C}^T(\mathbf{C}\mathbf{K}\mathbf{C}^T + \mathbf{R}_\mathbf{v})^{-1}\mathbf{C}\mathbf{K}\mathbf{\Phi}^T + \mathbf{R}_\omega - \mathbf{K} = \mathbf{0} \tag{2.5}$$

The minimum of the cost function J is then calculated by (see [12] for details)

$$J_{\min} = \mathrm{tr}\{\mathbf{Q}\mathbf{K}\}$$

Clearly, relation (2.4) provides us a solution to the optimal noise rejection problem via GSHF control, in the case where the hold function $\mathbf{F}(t)$ does not have a prespecified structure. Our attention is next focused on the special class of the time-varying T_0-periodic matrix functions $\mathbf{F}(t)$, for which every element of $\mathbf{F}(t)$, denoted by $f_{ij}(t)$, is piecewise constant over intervals of length $T_i = T_0/N_i$, with $N_i \in \mathbb{Z}^+$, i.e.

$$f_{ij}(t) = f_{ij,\mu},\ \forall t \in [\mu T_i, (\mu+1)T_i)$$

for $\mu = 0, \ldots, N_i - 1$. In this case, as it has been shown in [16], the following relation holds

$$\tilde{\mathbf{F}}_\phi = \hat{\mathbf{B}}\hat{\mathbf{F}} \tag{2.6}$$

where, defining by $\mathbf{b}_i$ the ith column of $\mathbf{B}$,

$$\hat{\mathbf{B}} = \lfloor \hat{\mathbf{b}}_1 \cdots \hat{\mathbf{A}}_1^{N_1-1}\hat{\mathbf{b}}_1 \cdots \hat{\mathbf{b}}_m \cdots \hat{\mathbf{A}}_m^{N_m-1}\hat{\mathbf{b}}_m \rfloor,$$

$$\hat{\mathbf{A}} \,\hat{=}\, \exp(\mathbf{A}T_i),\ \hat{\mathbf{b}}_i \,\hat{=} \int_0^{T_i} \exp(\mathbf{A}\lambda)\mathbf{b}_i d\lambda$$

and where the $m \times p$ block matrix $\hat{\mathbf{F}}$ has the form

$$\hat{\mathbf{F}} = \begin{bmatrix} \hat{\mathbf{f}}_{11} & \cdots & \hat{\mathbf{f}}_{1p} \\ \vdots & \ddots & \vdots \\ \hat{\mathbf{f}}_{m1} & \cdots & \hat{\mathbf{f}}_{mp} \end{bmatrix}, \quad \hat{\mathbf{f}}_{\mathrm{ij}} = \begin{bmatrix} f_{ij,N_i-1} \\ \vdots \\ f_{ij,0} \end{bmatrix}$$

In this case, ith row $\mathbf{f}_i^T(t)$ of the matrix $\mathbf{F}(t)$ and the ith block row of the matrix $\hat{\mathbf{F}}$ are interrelated as

$$\mathbf{f}_i^T(t) = [f_{i1}(t) \quad \cdots \quad f_{ip}(t)] = \mathbf{e}_{N_i-\mu}\lfloor \hat{\mathbf{f}}_{i1} \cdots \hat{\mathbf{f}}_{ip} \rfloor$$

$\forall\ \mu T_i \leqslant t < (\mu+1)T_i$ for $i = 1, \ldots, m$ and for $\mu = 0, \ldots,\ N_i$-1, where $\mathbf{e}_{N_i-\mu} \in \mathbb{R}^{N_i}$ is the row vector, whose elements are zero except for a unity appearing in the $(N_i - \mu)$th position.

From the above analysis it becomes clear that in the case where the elements of $\mathbf{F}(t)$, are piecewise constant, the admissible hold function can be obtained on the basis of $\hat{\mathbf{F}}$. To find matrix $\hat{\mathbf{F}}$ one must solve (2.6). Note that (2.6) is always solvable if $N_i \geqslant n_i$, $i = 1, 2, \ldots, m$, where n_i comprise a set of locally minimum controllability indices of the pair $(\mathbf{A},\mathbf{B})$ (for details see [16]).

3. Stability robustness of the optimal GSHF based regulator

In this Section, our aim will be the study of the robustness properties of the above GSHF based regulator, designed in order to achieve optimal noise rejection. Our concern, in this Section, is to analyze, how sensitive the stable modes of the closed-loop system will be under small variations of the plant parameters and in particular whether these modes will remain inside the unit circle for such variations. Our investigation allows suggesting lower bounds for the minimum singular value of the regulator's return difference matrix. On the basis of these bounds, guaranteed stability margins for GSHF based optimal regulators, are established, as measures of its stability robustness. The proposed stability margins are obtained on the basis of a fundamental spectral factorization equality, called the GSHF Return Difference Equality, and are expressed explicitly in terms of the elementary cost and system matrices. It is worth noticing, at this point, that our investigation on guaranteed stability margins of the GSHF based optimal regulator, is focused on a variety of important particular cases, for the elementary system and cost matrices. The reason for such a type of investigation, is due to the fact that, in our case, it is very difficult (if not impossible) to obtain a universal lower bound for the minimum singular value of the regulator's return difference matrix, as in the case of the continuous-time LQ regulator [23]. This difficulty stems from the entanglement of the solution of the Riccati equation, in the left hand side of the GSHF Return Difference Equality.

In order to present our results, we first recall some well-known results from optimal LQ regulation theory. More precisely, it is well known that the eigenvalues of the matrix $\mathbf{\Phi} + \tilde{\mathbf{F}}_\phi \mathbf{C}$ are the zeros of the return difference matrix $\mathbf{\Omega}(z)$ of the form

$$\mathbf{\Omega}(z) = \mathbf{I} - \mathbf{T}(z) \equiv \mathbf{I} - \mathbf{C}(z\mathbf{I} - \mathbf{\Phi})^{-1}\tilde{\mathbf{F}}_\phi$$

at the plant input, where $\mathbf{T}(z)$ is the loop transfer function of the plant. Matrix $\mathbf{\Omega}(z)$ satisfies the following fundamental spectral factorization equality, called the *GSHF* Return Difference Equality

$$\mathbf{\Omega}^T(z^{-1})(\mathbf{R_v} + \mathbf{CKC}^T)\mathbf{\Omega}(z) = \mathbf{R_v} + \mathbf{C}(z^{-1}\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{R}_\omega(z\mathbf{I} - \mathbf{\Phi}^T)^{-1}\mathbf{C}^T$$

For single-input, single-output systems, stability margins are commonly used measures of the robustness of a feedback loop and can be very easily determined using Bode or Nyquist diagrams. For multi-input, multi-output systems, stability margins, for unstructured additive perturbations of the closed-loop, can be obtained on the basis of the minimum singular value of the return difference matrix on the stability boundary. The minimum inward and upward gain margins of the GSHF based optimal regulator are defined, in general, to be the positive scalars GM^{in} and GM^{up} for which a simultaneous insertion of gains g_i, $i = 1, 2 \ldots, m$, in the ith feedback loop of the closed-loop regulator will not destabilize the closed-loop system if

$$GM^{\mathrm{in}} \leqslant g_i \leqslant GM^{\mathrm{up}}$$

Similarly, the guaranteed phase margin of the regulator is defined to be the scalar PM for which a simultaneous insertion of the phase factor $e^{j\varphi_i}$, $i = 1, 2, \ldots, m$, in the above ith feedback loop will keep the closed-loop stable if

$$|\varphi_i| \leqslant PM$$

In the sequel, let $\sigma_{\max}(\mathbf{M})$ and $\sigma_{\min}(\mathbf{M})$ be the maximum and the minimum singular values of a matrix $\mathbf{M}$, respectively. Then, the following Proposition has been proven in [12,21].

PROPOSITION 3.1. Consider a stable feedback system with loop transfer function $\mathbf{T}(z)$ and return difference $\mathbf{\Omega}(z)$ (where we assume positive feedback). Suppose $\Delta\mathbf{T}(z)$ undergoes an additive change $\Delta\mathbf{T}(z)$ which preserves the number of unstable poles of $\mathbf{T}(z)$. Then:

Stability is preserved if

$$\sigma_{\max}[\Delta\mathbf{T}(z)] \leqslant \sigma_{\min}[\mathbf{\Omega}(z)],$$

whenever $|z| = 1$

If there exists $\beta \in [0,1)$ such that $\sigma[\mathbf{\Omega}(z)] \geqslant \beta$ for $|z| = 1$, then multivariable gain and phase margins are

$$GM^{\text{in}} = (1+\beta)^{-1}, GM^{\text{up}} = (1-\beta)^{-1}, PM = \pm\arccos\left(1-\frac{\beta^2}{2}\right) \tag{3.1}$$

On the basis of Proposition 3.1, we can establish the following Theorem.

THEOREM 3.1. Suppose that the hypotheses of Proposition 3.1 are satisfied. Suppose also that matrix $\mathbf{\Phi}$ satisfies one of the following properties:

(a) Matrix $\mathbf{\Phi}$ is asymptotically stable and $\sigma_{\max}(\mathbf{\Phi}) < 1$.

(b) Matrix $\mathbf{\Phi}$ is asymptotically stable, its eigenvalues are all distinct and $\sigma_{\max}(\mathbf{\Phi}) \geqslant 1$.

The eigenvalues of $\mathbf{\Phi}$ are all distinct, none of them lies on the unit circle, but some of them lie outside this circle. It is further assumed that none of the latter is a reciprocal of the remaining eigenvalues.

Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z| = 1$, by α_Ω, where

$$\alpha_\Omega^2 = \frac{\sigma_{\min}(\mathbf{R}_\mathbf{v})}{\sigma_{\max}(\mathbf{R}_\mathrm{V}) + \sigma_{\max}^2(\mathbf{C})\delta},$$

$$\delta = \begin{cases} \sigma_{\max}(\mathbf{R}_\omega)[1-\sigma_{\max}^2(\mathbf{\Phi})]^{-1}, & \text{for Case (a)} \\ \sigma_{\max}(\mathbf{R}_\omega)[1-\nu_{\max}^2(\mathbf{\Phi})]^{-1}, & \text{for Case (b)} \\ \dfrac{\sigma_{\max}(\mathbf{R}_\omega)+\omega^2}{1-\hat{\rho}^2}, & \text{for Case (c)} \end{cases} \tag{3.2}$$

The scalar $\nu_{\max}(\mathbf{\Phi})$ is the maximum absolute value of the eigenvalues of $\mathbf{\Phi}$, and ω is defined as

$$\omega = \frac{\sigma_{\max}(\mathbf{V}_{\text{out}})\sigma_{\max}(\mathbf{V}_{\text{out}}\mathbf{C}^T)\sigma_{\max}(\mathbf{R}_\mathbf{v})[\nu_{\max}^2(\mathbf{\Phi})-1]}{\sigma_{\min}(\Phi_{\text{out}})\sigma_{\min}^{1/2}(\mathbf{R}_\mathbf{v})\sigma_{\min}^2(\mathbf{C}\hat{\mathbf{V}}_{\text{out}}^+)}$$

where $\mathbf{\Phi}_{\text{out}}$ and $\mathbf{V}_{\text{out}}$ are the diagonal Jordan block of the outside the unit circle eigenvalues of $\mathbf{\Phi}$ and the matrix of their coresponding eigenrows, respectively, and $\mathbf{V}_{\text{out}}^+$ is the conjugate transpose of $\mathbf{V}_{\text{out}}$. Finally, the scalar $\hat{\rho}$ is defined, in the third case, as

$$\hat{\rho} \triangleq \max\left\{\rho^*, \frac{1}{\rho^{**}}\right\}$$

where ρ^* is the largest absolute value of the eigenvalues of $\mathbf{\Phi}$ inside the unit circle and ρ^{**} is the smallest absolute value of the remaining eigenvalues of $\mathbf{\Phi}$. Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta = \alpha_\mathbf{\Omega}$.

Proof: From the results in [12,21] we have

$$\sigma^2_{\min}[\mathbf{\Omega}(z)] \geqslant \frac{\sigma_{\min}(\mathbf{R_v})}{\sigma_{\max}(\mathbf{R_v} + \mathbf{CKC}^T)}, \quad \text{for } |z| = 1$$

Using simple singular value properties we obtain

$$\sigma^2_{\min}[\mathbf{\Omega}(z)] \geqslant \frac{\sigma_{\min}(\mathbf{R_v})}{\sigma_{\max}(\mathbf{R_v}) + \sigma^2_{\max}(\mathbf{C})\sigma_{\max}(\mathbf{K})}, \quad \text{for } |z| = 1 \tag{3.3}$$

Using an argument analogous to that reported in [21], we can easily conclude that

$$\sigma_{\max}(\mathbf{K}) \leqslant \delta$$

where δ is given by (3.2). Therefore, for $|z| = 1$, $\sigma_{\min}[\mathbf{\Omega}(z)]$ is bounded from below by $\alpha_{\mathbf{\Omega}}$ defined by (3.2). Moreover, since $0 < \alpha_{\mathbf{\Omega}} < 1$, (3.1) can be applied with $\beta = \alpha_{\mathbf{\Omega}}$.

We are also able to establish the following result.

THEOREM 3.2. Suppose that matrix $\hat{\mathbf{R}}_\mathbf{v}$ of the form

$$\hat{\mathbf{R}}_\mathbf{v} = \mathbf{C}^T\mathbf{R}_\mathbf{v}^{-1}\mathbf{C}$$

is nonsingular. Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z| = 1$, by $\beta_{\mathbf{\Omega}}$, where

$$\beta^2_{\mathbf{\Omega}} = \frac{\sigma_{\min}(\mathbf{R_v})}{\sigma_{\max}(\mathbf{R_v}) + \sigma^2_{\max}(\mathbf{C})\vartheta}, \quad \vartheta = \frac{\sigma^2_{\max}(\mathbf{\Phi})\sigma_{\max}(\mathbf{R_v})}{\sigma^2_{\min}(\mathbf{C})} + \sigma_{\max}(\mathbf{R}_\omega)$$

Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta = \beta_{\mathbf{\Omega}}$.

Proof: Observe that equation (2.5) can also be written as

$$\mathbf{K} = \mathbf{\Phi K \Phi}^T + \mathbf{R}_\omega - \mathbf{\Phi K C}_R^T(\mathbf{I} + \mathbf{C}_R\mathbf{K C}_R^T)^{-1}\mathbf{C}_R\mathbf{K \Phi}^T$$

where

$$\mathbf{C}_R = \mathbf{R}_\mathbf{v}^{-\frac{1}{2}}\mathbf{C}$$

Suppose now that matrix $\hat{\mathbf{R}}_\mathbf{v}$ is nonsingular. Then, from the results in [24] we have

$$\mathbf{K} \leqslant \mathbf{\Phi}\hat{\mathbf{R}}_\mathbf{v}^{-1}\mathbf{\Phi}^T + \mathbf{R}_\omega$$

Therefore,

$$\begin{aligned}\sigma_{\max}(\mathbf{K}) &\leqslant \sigma_{\max}(\mathbf{\Phi}\hat{\mathbf{R}}_{\mathbf{v}}^{-1}\mathbf{\Phi}^T + \mathbf{R}_{\omega}) \leqslant \sigma_{\max}^2(\mathbf{\Phi})\sigma_{\max}(\hat{\mathbf{R}}_{\mathbf{v}}^{-1}) + \sigma_{\max}(\mathbf{R}_{\omega}) \\ &\leqslant \frac{\sigma_{\max}^2(\mathbf{\Phi})}{\sigma_{\min}(\hat{\mathbf{R}}_{\mathbf{v}})} + \sigma_{\max}(\mathbf{R}_{\omega}) = \frac{\sigma_{\max}^2(\mathbf{\Phi})}{\sigma_{\min}(\mathbf{C}^T\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{C})} + \sigma_{\max}(\mathbf{R}_{\omega}) \\ &\leqslant \frac{\sigma_{\max}^2(\mathbf{\Phi})}{\sigma_{\min}^2(\mathbf{C})\sigma_{\min}(\mathbf{R}_{\mathbf{v}}^{-1})} + \sigma_{\max}(\mathbf{R}_{\omega}) = \frac{\sigma_{\max}^2(\mathbf{\Phi})\sigma_{\max}(\mathbf{R}_{\mathbf{v}})}{\sigma_{\min}^2(\mathbf{C})} + \sigma_{\max}(\mathbf{R}_{\omega})\end{aligned}$$

Hence $\sigma_{\max}(\mathbf{K}) \leqslant \vartheta$, $\sigma_{\min}[\mathbf{\Omega}(z)] \geqslant \beta_{\mathbf{\Omega}}$ and since $0 < \beta_{\mathbf{\Omega}} < 1$, (3.1) can be applied with $\beta = \beta_{\mathbf{\Omega}}$.

In the case where $\hat{\mathbf{R}}_{\mathbf{v}}$ is singular, guaranteed stability margins of the GSHF based optimal regulator can be obtained as suggested by the following Theorem.

THEOREM 3.3. Suppose that

$$\mathbf{R}_{\omega} > \mathbf{0} \quad \text{and} \quad \sigma_{\max}^2(\mathbf{\Phi}) < 1 + \sigma_{\min}^2(\mathbf{C}_R)\eta, \;\; \eta = \lambda_{\max}[\mathbf{\Phi}[\mathbf{R}_{\omega}^{-1} + \hat{\mathbf{R}}_{\mathbf{v}}]^{-1}\mathbf{\Phi}^T + \mathbf{R}_{\omega}] \tag{3.4}$$

Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z| = 1$, by $\gamma_{\mathbf{\Omega}}$, where

$$\gamma_{\mathbf{\Omega}}^2 = \frac{\sigma_{\min}(\mathbf{R}_{\mathbf{v}})}{\sigma_{\max}(\mathbf{R}_{\mathbf{v}}) + \sigma_{\max}^2(\mathbf{C})\mu},$$

$$\mu = \frac{\lambda_{\max}(\mathbf{R}_{\omega})}{1 + \sigma_{\min}^2(\mathbf{C}_R)\eta - \sigma_{\max}^2(\mathbf{\Phi})}\,\sigma_{\max}^2(\mathbf{\Phi}) + \sigma_{\max}(\mathbf{R}_{\omega})$$

Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta = \gamma_{\mathbf{\Omega}}$.

Proof: If (3.4) holds, then $\mathbf{K}$ obeys the following inequality [25]

$$\mathbf{K} \leqslant \frac{\lambda_{\max}(\mathbf{R}_{\omega})}{1 + \sigma_{\min}^2(\mathbf{C}_R)\eta - \sigma_{\max}^2(\mathbf{\Phi})}\,\mathbf{\Phi}\mathbf{\Phi}^T + \mathbf{R}_{\omega}$$

Therefore,

$$\begin{aligned}\sigma_{\max}(\mathbf{K}) &\leqslant \sigma_{\max}\left(\frac{\lambda_{\max}(\mathbf{R}_{\omega})}{1 + \sigma_{\min}^2(\mathbf{C}_{\mathrm{R}})\eta - \sigma_{\max}^2(\mathbf{\Phi})}\,\mathbf{\Phi}\mathbf{\Phi}^T + \mathbf{R}_{\omega}\right) \\ &\leqslant \frac{\lambda_{\max}(\mathbf{R}_{\omega})}{1 + \sigma_{\min}^2(\mathbf{C}_{\mathrm{R}})\eta - \sigma_{\max}^2(\mathbf{\Phi})}\sigma_{\max}^2(\mathbf{\Gamma}) + \sigma_{\max}(\mathbf{R}_{\omega}) = \mu\end{aligned} \tag{3.5}$$

On the basis of (3.5) we obtain $\sigma_{\min}[\mathbf{\Omega}(z)] \geqslant \gamma_{\mathbf{\Omega}}$ and since $0 < \gamma_{\mathbf{\Omega}} < 1$, (3.1) can be applied with $\beta = \gamma_{\mathbf{\Omega}}$.

In the case where it is impossible to obtain the bound $\gamma_{\mathbf{\Omega}}$, due to the fact that inequalities (3.4) do not hold, the following Theorem provides us guaranteed stability margins for the GSHF based optimal regulator.

THEOREM 3.4. Suppose that

$$\lambda_{\max}[\mathbf{\Phi}(\mathbf{I}+\hat{\eta}\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T]<1,\ \hat{\eta}=\lambda_{\max}[\mathbf{\Phi}(\Pi^{-1}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T+\mathbf{R}_\omega] \tag{3.6}$$

$$\mathbf{\Pi}=\mathbf{\Phi}(\xi^{-1}\mathbf{I}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T+\mathbf{R}_\omega$$

$$\xi=\frac{\sigma^2_{\min}(\mathbf{\Phi})+\sigma^2_{\max}(\mathbf{C}_R)\lambda_{\min}(\mathbf{R}_\omega)-1+\sqrt{[\sigma^2_{\min}(\mathbf{\Phi})+\sigma^2_{\max}(\mathbf{C}_R)\lambda_{\min}(\mathbf{R}_\omega)-1]^2+4\sigma^2_{\max}(\mathbf{C}_R)\lambda_{\min}(\mathbf{R}_\omega)}}{2\sigma^2_{\max}(\mathbf{C}_R)}$$

Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z|=1$, by $\delta_{\mathbf{\Omega}}$, where

$$\delta^2_{\mathbf{\Omega}}=\frac{\sigma_{\min}(\mathbf{R}_{\mathbf{v}})}{\sigma_{\max}(\mathbf{R}_{\mathbf{v}})+\sigma^2_{\max}(\mathbf{C})\hat{\mu}},\quad \hat{\mu}=\sigma_{\max}(\mathbf{\Phi}(\mathbf{N}^{-1}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T)+\sigma_{\max}(\mathbf{R}_\omega)$$

$$\mathbf{N}=\mathbf{\Phi}(\kappa^{-1}\mathbf{I}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T+\mathbf{R}_\omega,\ \kappa=\{1-\lambda_{\max}[\mathbf{\Phi}(\mathbf{I}+\hat{\eta}\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T]\}^{-1}\lambda_{\max}(\mathbf{R}_\omega)$$

Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta=\delta_{\mathbf{\Omega}}$.

Proof: If (3.6) holds, then, according to the results reported in [26], the following upper bound can be obtained for $\mathbf{K}$

$$\mathbf{K}\leqslant\mathbf{\Phi}(\mathbf{N}^{-1}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T+\mathbf{R}_\omega$$

Therefore,

$$\begin{aligned}\sigma_{\max}(\mathbf{K})&\leqslant\sigma_{\max}(\mathbf{\Phi}(\mathbf{N}^{-1}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T+\mathbf{R}_\omega)\\&\leqslant\sigma_{\max}(\mathbf{\Phi}(\mathbf{N}^{-1}+\hat{\mathbf{R}}_{\mathbf{v}})^{-1}\mathbf{\Phi}^T)+\sigma_{\max}(\mathbf{R}_\omega)=\hat{\mu}\end{aligned} \tag{3.7}$$

On the basis of (3.7) we obtain $\sigma_{\min}[\mathbf{\Omega}(z)]\geqslant\delta_{\mathbf{\Omega}}$ and since $0<\delta_{\mathbf{\Omega}}<1$, (3.1) can be applied with $\beta=\delta_{\mathbf{\Omega}}$.

In the particular case where matrix $\mathbf{\Phi}$ is asymptotically stable, guaranteed stability margins for the GSHF based optimal regulator can be obtained as suggested by the following three Theorems.

THEOREM 3.5. Let $\mathbf{\Phi}$ be an asymptotically stable matrix with $\sigma_{\max}(\mathbf{\Phi})<1$. Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z|=1$, by $\epsilon_{\mathbf{\Omega}}$, where

$$\epsilon^2_{\mathbf{\Omega}}=\frac{\sigma_{\min}(\mathbf{R}_{\mathbf{v}})}{\sigma_{\max}(\mathbf{R}_{\mathbf{v}})+\sigma^2_{\max}(\mathbf{C})\pi},\ \pi=\frac{\lambda_{\max}(\mathbf{R}_\omega)}{1-\sigma^2_{\max}(\mathbf{\Phi})}\sigma^2_{\max}(\mathbf{\Phi})+\sigma_{\max}(\mathbf{R}_\omega)$$

Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta=\epsilon_{\mathbf{\Omega}}$.

Proof: Let $\mathbf{K}_L$ be the positive definite solution of the following Lyapunov equation

$$\mathbf{K}_L=\mathbf{\Phi}\mathbf{K}_L\mathbf{\Phi}^T+\mathbf{R}_\omega$$

If matrix $\mathbf{\Phi}$ is asymptotically stable, then, it is evident that $\mathbf{0} \leqslant \mathbf{K} \leqslant \mathbf{K}_L$. Therefore,

$$\sigma_{\max}(\mathbf{K}) \leqslant \sigma_{\max}(\mathbf{K}_L) \tag{3.8}$$

If $\sigma_{\max}(\mathbf{\Phi}) < 1$, then according to the results in [27] we have

$$\mathbf{K}_L \leqslant \frac{\lambda_{\max}(\mathbf{R}_\omega)}{1 - \sigma^2_{\max}(\mathbf{\Phi})} \mathbf{\Phi}\mathbf{\Phi}^T + \mathbf{R}_\omega$$

Therefore,

$$\begin{aligned}\sigma_{\max}(\mathbf{K}_L) &\leqslant \sigma_{\max}\Big[\frac{\lambda_{\max}(\mathbf{R}_\omega)}{1 - \sigma^2_{\max}(\mathbf{\Phi})} \mathbf{\Phi}\mathbf{\Phi}^T + \mathbf{R}_\omega\Big] \\ &\leqslant \frac{\lambda_{\max}(\mathbf{R}_\omega)}{1 - \sigma^2_{\max}(\mathbf{\Phi})} \sigma_{\max}(\mathbf{\Phi}\mathbf{\Phi}^T) + \sigma_{\max}(\mathbf{R}_\omega) \\ &\leqslant \frac{\lambda_{\max}(\mathbf{R}_\omega)}{1 - \sigma^2_{\max}(\mathbf{\Phi})} \sigma^2_{\max}(\mathbf{\Phi}) + \sigma_{\max}(\mathbf{R}_\omega) = \pi\end{aligned}$$

Hence, $\sigma_{\max}(\mathbf{K}) \leqslant \pi$ and $\sigma_{\min}[\mathbf{\Omega}(z)] \geqslant \epsilon_{\mathbf{\Omega}}$. Then, since $0 < \epsilon_{\mathbf{\Omega}} < 1$, (3.1) can be applied with $\beta = \epsilon_{\mathbf{\Omega}}$.

THEOREM 3.6. Let $\mathbf{\Phi}$ be asymptotically stable with distinct eigenvalues. Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z| = 1$, by $\zeta_{\mathbf{\Omega}}$, where

$$\zeta^2_{\mathbf{\Omega}} = \frac{\sigma_{\min}(\mathbf{R}_\mathbf{v})}{\sigma_{\max}(\mathbf{R}_\mathbf{v}) + \sigma^2_{\max}(\mathbf{C})\phi}, \quad \phi = \frac{\lambda_{\max}(\mathbf{M}^T\mathbf{R}_\omega\mathbf{M})}{1 - \nu^2_{\max}(\mathbf{\Phi})} \sigma^{-2}_{\min}(\mathbf{M})$$

and $\mathbf{M}$ is the nonsingular permutation matrix, defined as

$$\mathbf{M}^{-1}\mathbf{\Phi}^T\mathbf{M} = \mathbf{L}, \quad \mathbf{L} = \mathrm{diag}\{\lambda_i(\mathbf{\Phi})\},\ i = 1,2,\ldots,n$$

Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta = \zeta_{\mathbf{\Omega}}$.

Proof: If matrix $\mathbf{\Phi}$ is asymptotically stable and diagonalizable, then one can obtain (see [28] for details)

$$\mathbf{K}_L \leqslant \lambda_{\max}(\mathbf{M}^T\mathbf{R}_\omega\mathbf{M})(\mathbf{M}^{-1})^T \underset{i=1,2,\ldots,n}{\mathrm{diag}} \left\{\frac{1}{1 - |\lambda_i(\mathbf{\Phi})|^2}\right\} \mathbf{M}^{-1}$$

Therefore,

$$\begin{aligned}
\sigma_{\max}(\mathbf{K}_L) &\leqslant \sigma_{\max}\left[\lambda_{\max}(\mathbf{M}^T\mathbf{R}_\omega\mathbf{M})(\mathbf{M}^{-1})^T \operatorname*{diag}_{i=1,2,\ldots,n}\left\{\frac{1}{1-|\lambda_i(\mathbf{\Phi})|^2}\right\}\mathbf{M}^{-1}\right] \\
&= \lambda_{\max}(\mathbf{M}^T\mathbf{R}_\omega\mathbf{M})\sigma_{\max}\left[(\mathbf{M}^{-1})^T \operatorname*{diag}_{i=1,2,\ldots,n}\left\{\frac{1}{1-|\lambda_i(\mathbf{\Phi})|^2}\right\}\mathbf{M}^{-1}\right] \\
&\leqslant \lambda_{\max}(\mathbf{M}^T\mathbf{R}_\omega\mathbf{M})\sigma^2_{\max}(\mathbf{M}^{-1})\sigma_{\max}\left[\operatorname*{diag}_{i=1,2,\ldots,n}\left\{\frac{1}{1-|\lambda_i(\mathbf{\Phi})|^2}\right\}\right] \\
&= \frac{\lambda_{\max}(\mathbf{M}^T\mathbf{R}_\omega\mathbf{M})}{1-\nu^2_{\max}(\mathbf{\Phi})}\sigma^{-2}_{\min}(\mathbf{M}) = \phi \qquad (3.9)
\end{aligned}$$

Combining (3.8) and (3.9) yields $\sigma_{\max}(\mathbf{K}) \leqslant \phi$. Therefore, $\sigma_{\min}[\mathbf{\Omega}(z)] \geqslant \zeta_{\mathbf{\Omega}}$, and since $0 < \zeta_{\mathbf{\Omega}} < 1$, (3.1) can be applied with $\beta = \zeta_{\mathbf{\Omega}}$.

THEOREM 3.7. Suppose that $\mathbf{\Phi}$ is an asymptotically stable and normal matrix. Then, the minimum singular value of $\mathbf{\Omega}(z)$ is bounded from below, for $|z| = 1$, by $\eta_{\mathbf{\Omega}}$, where

$$\eta^2_{\mathbf{\Omega}} = \frac{\sigma_{\min}(\mathbf{R}_\mathbf{v})}{\sigma_{\max}(\mathbf{R}_\mathbf{v}) + \sigma^2_{\max}(\mathbf{C})\psi}, \quad \psi = \frac{\lambda_{\max}(\mathbf{R}_\omega)}{\sigma_{\min}(\mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T)}$$

Moreover, the guaranteed gain and phase margins of the GSHF based optimal regulator are then obtained by (3.1) with $\beta = \eta_{\mathbf{\Omega}}$.

Proof: If $\mathbf{\Phi}$ is asymptotically stable and normal, then according to the results in [28] we have

$$\mathbf{K}_L \leqslant \lambda_{\max}(\mathbf{R}_\omega)(\mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T)^{-1}$$

Therefore,

$$\begin{aligned}
\sigma_{\max}(\mathbf{K}_L) &\leqslant \sigma_{\max}[\lambda_{\max}(\mathbf{R}_\omega)(\mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T)^{-1}] = \lambda_{\max}(\mathbf{R}_\omega)\sigma_{\max}[(\mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T)^{-1}] \\
&= \frac{\lambda_{\max}(\mathbf{R}_\omega)}{\sigma_{\min}(\mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T)} = \psi \qquad (3.10)
\end{aligned}$$

Combining (3.8) and (3.10) yields $\sigma_{\max}(\mathbf{K}) \leqslant \psi$. Therefore, $\sigma_{\min}[\mathbf{\Omega}(z)] \geqslant \eta_{\mathbf{\Omega}}$, and since $0 < \eta_{\mathbf{\Omega}} < 1$, (3.1) can be applied with $\beta = \eta_{\mathbf{\Omega}}$.

4. Mathematical modeling of civil engineering structures

As already mentioned, in the present paper the optimal noise rejection problem of civil engineering structures is investigated using GSHF. In order to fulfil our investigation, the state space model of our structure is formulated, first, without control as

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbb{C}\dot{\mathrm{d}}(t) + \mathbf{K_S}\mathbf{d}(t) = \boldsymbol{q}(t) \qquad (4.1)$$

where $\mathbf{M} \in \mathbb{R}^{q\times q}$ is a mass matrix, $\mathbb{C} \in \mathbb{R}^{q\times q}$ is a damping matrix, $\mathbf{K_S} \in \mathbb{R}^{q\times q}$ is a stiffness matrix, $\mathbf{q}(t) \in \mathbb{R}^{q}$ is a load vector, $\mathbf{d}(t) \in \mathbb{R}^{q}$ is a displacement vector, $\dot{\mathbf{d}}(t) \in \mathbb{R}^{q}$ is a velocity vector and $\ddot{\mathbf{d}}(t) \in \mathbb{R}^{q}$ is an acceleration vector. Using the substitution $\dot{\mathbf{d}}(t) = \mathbf{g}(t)$, in the presence of control forces $\mathbf{u}(t) \in \mathbb{R}^{m}$**,** relations (4.1) can be rewritten as

$$\mathbf{M}\dot{\mathbf{g}}(t) + \mathbb{C}\mathbf{g}(t) + \mathbf{K_S}\mathbf{d}(t) = \boldsymbol{q}(t) + \mathbf{B}_0\mathbf{u}(t),\ \dot{\mathbf{d}}(t) - \mathbf{g}(t) = \mathbf{0} \tag{4.2}$$

where, $\mathbf{B}_0 \in \mathbf{R}^{q\times m}$ is a control forces arrangement matrix. Relations (4.2) can be written in a compact matrix form as

$$\begin{bmatrix} \dot{\mathbf{d}}(t) \\ \dot{\mathbf{g}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{q\times q} & \mathbf{I}_{q\times q} \\ -\mathbf{M}^{-1}\mathbf{K_S} & -\mathbf{M}^{-1}\mathbb{C} \end{bmatrix} \begin{bmatrix} \mathbf{d}(t) \\ \mathbf{g}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{q\times m} \\ \mathbf{M}^{-1}\mathbf{B}_0 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} \mathbf{0}_{q\times q} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{q}(t) \tag{4.3}$$

Relation (4.3), may further be written in a standard matrix-vector state space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}\mathbf{q}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \tag{4.4a}$$

where, for $n = 2q$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{D} \in \mathbb{R}^{n\times q}$, $\mathbf{x}(t) \in \mathbb{R}^{n}$ and $\mathbf{w}(t) \in \mathbb{R}^{n}$ have the following forms

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{q\times q} & \mathbf{I}_{q\times q} \\ -\mathbf{M}^{-1}\mathbf{K_S} & -\mathbf{M}^{-1}\mathbb{C} \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} \mathbf{0}_{q\times m} \\ \mathbf{M}^{-1}\mathbf{B}_0 \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} \mathbf{0}_{q\times q} \\ \mathbf{M}^{-1} \end{bmatrix},$$
$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{d}(t) \\ \mathbf{g}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{d}(t) \\ \dot{\mathbf{d}}(t) \end{bmatrix},\ \mathbf{w}(t) = \mathbf{D}q(t) \tag{4.4b}$$

We further assume that the p outputs $\mathbf{y}(t) \in \mathbb{R}^{p}$ of the structure are a combination of some, say $r_1 \leqslant q$, of the displacements and some, say $r_2 \leqslant q$, of the velocities. In this case, we have $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, where

$$\mathbf{C} = \begin{bmatrix} \mathbf{e}_{j_1} & \\ \mathbf{e}_{j_2} & \mathbf{0}_{r_1\times q} \\ \vdots & \\ \mathbf{e}_{j_{r_1}} & \\ & \mathbf{e}_{i_1} \\ & \mathbf{e}_{i_2} \\ \mathbf{0}_{r_2\times q} & \vdots \\ & \mathbf{e}_{i_{r_2}} \end{bmatrix} \tag{4.4c}$$

and where, $j_1,\ j_2, \ldots,\ j_{r_1}$ are the indices of the particular displacements of interest while $i_1,\ i_2, \cdots,\ i_{r_2}$ are the indices of the particular velocities of interest.

5. Simulation study of the proposed technique

The IPB 800 three-story, single bay, steel frame structure under dynamic loading $\mathbf{q}(t)$ [6] is examined in this Section, as an example of the application of the technique presented in the previous Sections. For the structure studied here, the mass of each floor is $m_i = 16$ t, $i = 1,2,3$. The other characteristics of the structure are $\tilde{F}_i = 670.13$ t/m and $h_i = 3$ m, i = 1,2,3, $\mathbf{B}_0 = \mathbf{I}_{3\times 3}$, and

$$\mathbb{C} = \begin{bmatrix} 3.2 & 0 & 0 \\ 0 & 3.2 & 0 \\ 0 & 0 & 3.2 \end{bmatrix}.$$

With these values, the matrices of the state space description of the structure is obtained by (4.4), as

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -83.7663 & 83.7663 & 0 & -0.2 & 0 & 0 \\ 83.7663 & -167.5325 & 83.7663 & 0 & -0.2 & 0 \\ 0 & 83.7663 & -167.5325 & 0 & 0 & -0.2 \end{bmatrix},$$

$$\mathbf{B} = \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/16 & 0 & 0 \\ 0 & 1/16 & 0 \\ 0 & 0 & 1/16 \end{bmatrix}$$

We next apply to the above civil structure the GSHF based optimal regulation technique presented in Section 2. To this end, we focus our attention to the case where the outputs of the structure are the velocities of the three stories. In this case, we have $\mathbf{C} = [\mathbf{0}_{3\times 3} \quad \mathbf{I}_{3\times 3}]$. It can be easily checked that system with the above state space matrices is controllable and observable. Therefore, the technique presented in Section 2, is applicable in the present case. The covariance kernels of $\mathbf{w}(t)$ and $\mathbf{v}(kT_0)$ are

$$\mathbf{R}_{\mathbf{w}} = \begin{bmatrix} \mathbf{0}_{3\times 6} & \\ \mathbf{0}_{3\times 3} & 0.1 \times \mathbf{I}_{3\times 3} \end{bmatrix}, \mathbf{R}_{\mathbf{v}} = 0.002 \times \mathbf{I}_{3\times 3} > \mathbf{0} \tag{5.1}$$

Selecting $\mathbf{Q} = \mathbf{I}_{6\times 6}$, $T_0 = 2.5$ sec and applying the proposed technique in the case where $\mathbf{\Phi}(t)$ does not have a prespecified structure, we obtain

$$\mathbf{\Phi} = \begin{bmatrix} -0.6566 & 0.0683 & 0.0486 & -0.0792 & -0.0494 & -0.0276 \\ 0.0683 & 0.6763 & 0.0197 & -0.0494 & -0.0574 & -0.0218 \\ 0.0486 & 0.0197 & 0.7249 & -0.0276 & -0.0218 & -0.0298 \\ 2.4930 & -0.6666 & 0.4863 & -0.6407 & 0.0782 & 0.0541 \\ -0.6666 & 3.6459 & -1.1529 & 0.0782 & -0.6648 & 0.0241 \\ 0.4863 & -1.1529 & 3.1596 & 0.0541 & 0.0241 & -0.7189 \end{bmatrix}$$

$$\mathbf{K} = 10^{-4} \times \begin{bmatrix} 52 & 31 & 14 & -60 & -28 & -14 \\ & 35 & 17 & -28 & -46 & -14 \\ & & 20 & -14 & -14 & -32 \\ & \text{Symmetry} & & 1262 & 118 & 90 \\ & & & & 1234 & 28 \\ & & & & & 1144 \end{bmatrix},$$

$$\tilde{\mathbf{F}}_{\phi} = 10^{-4} \times \begin{bmatrix} 505 & 400 & 238 \\ 400 & 343 & 162 \\ 238 & 162 & 105 \\ 7334 & -505 & -253 \\ -505 & 7586 & -252 \\ -253 & -252 & 7839 \end{bmatrix}$$

Matrix $\tilde{\mathbf{F}}_{\phi}$ presents a certain kind of symmetry and the 3×3 regulator matrix $\mathbf{F}(t)$ is symmetric. In Figs. 1 and 2, the elements $f_{11}(t)$ and $f_{23}(t)$ of the admissible hold function $\mathbf{F}(t)$, are depicted.

In the case where our objective is to design hold functions with piecewise constant behavior, one must select N_I's and solve equation (2.6). For example, in the case where $N_1 = N_2 = N_3 = 15$, we obtain a symmetric multirate hold function $\mathbf{F}(t)$, whose elements $f_{11}(t)$ and $f_{23}(t)$ are depicted in Figs. 3 and 4. Similar results are obtained in the case where N_i's have different values. For example in the case where $N_1 = 30$, $N_2 = 45$ and $N_3 = 60$, the elements $f_{11}(t)$, $f_{23}(t)$ and $f_{32}(t)$ of the admissible multirate hold function are depicted in Figs. 5–7. Obviously, in this case the matrix $\mathbf{\Phi}(t)$ is not symmetric, due to different values of $\mathrm{N_i}$'s. Moreover, it is obvious that, as $N_i \to \infty$, the multirate GSHF obtained tends to the unconstrained GSHF depicted in Figs. 1, 2. Note that, in all the three cases presented above, the minimum of the cost function is

$$J_{\min} = tr\{\mathbf{KQ}\} = tr\{\mathbf{K}\} = 0.3747$$

Our concern, in the sequel, will be the study of the robustness properties of the

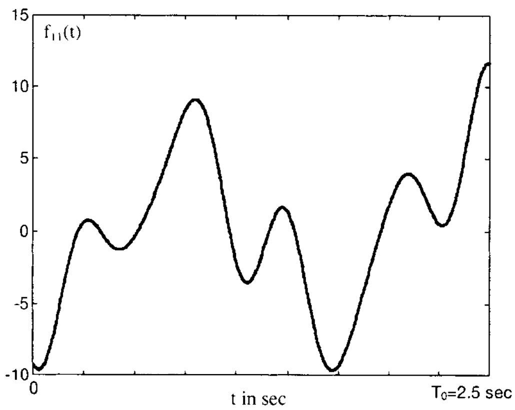

Figure 1. The 1-1 entry of $\mathbf{F}(t)$ in the case of an unconstrained GSHF and $T_0 = 2.5$ sec.

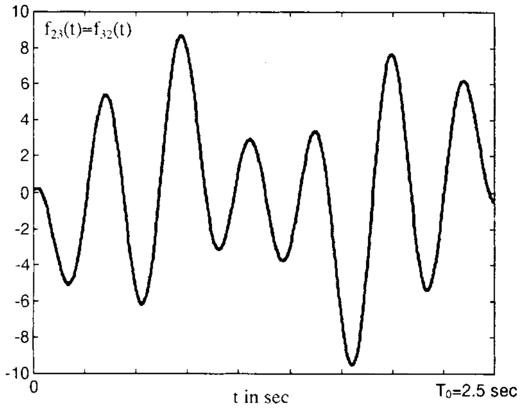

Figure 2. Entries 2-3 and 3-2 of $\mathbf{F}(t)$ in the case of an unconstrained GSHF and $T_0 = 2.5$ sec.

above designed GSHF based optimal regulator. To this end, the results presented in Section 3, are next applied, in order to obtain guaranteed stability margins for the regulator. In particular, in the present case, matrix $\mathbf{\Phi}$ is asymptotically stable, with distinct eigenvalues having the values $\lambda_{1,2} = -0.5671 \pm j0.5338$, $\lambda_{3,4} = -0.7533 \pm j0.1977$, $\lambda_{3,4} = -0.7207 \pm j0.2951$ and with $\sigma_{\max}(\mathbf{\Phi}) = 4.9712 > 1$. Therefore, in our case, Case (b) of Theorem 3.1 is applicable. On the other hand, Theorem 3.2 cannot be applied, since, in our case, matrix $\widehat{\mathbf{R}}_{\mathbf{v}}$ is not positive definite. Theorem 3.3 is applicable, since, in our case $\mathbf{R}_{\omega} > \mathbf{0}$ and $\sigma^2_{\max}(\mathbf{\Phi}) = 24.7124 < 1 + \sigma^2_{\min}(\mathbf{C}_R)\eta = 65.8997$ (note that $\eta = 0.1298$). Moreover, conditions of Theorem 3.6 are satisfied whereas conditions of Theorems 3.4 are not. Finally, Theorem 3.5 is not applicable, since $\sigma_{\max}(\mathbf{\Phi}) > 1$, while Theorem 3.7 is not applicable, since $\mathbf{\Phi}$ is not a normal matrix. Applying Theorems 3.1, 3.3 and 3.4, we obtain the bounds $\alpha_{\mathbf{\Omega}} =$

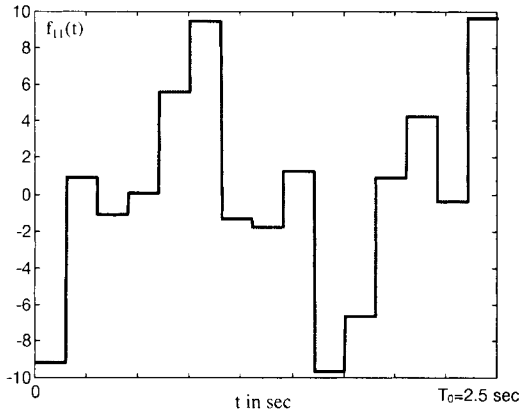

Figure 3. The 1-1 entry of $\mathbf{F}(t)$ in the case of a multirate GSHF with $N_1 = N_2 = N_3 = 15$ and $T_0 = 2.5$ sec.

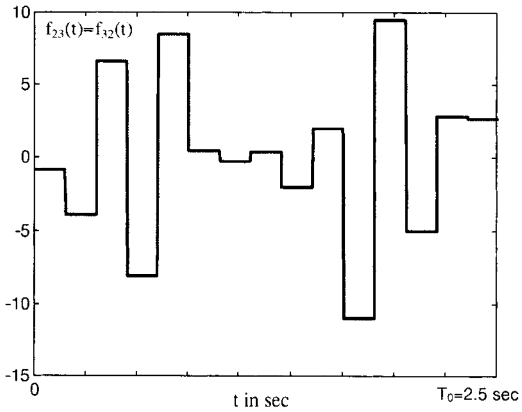

Figure 4. Entries 2-3 and 3-2 of $\mathbf{F}(t)$ in the case of a multirate GSHF with $N_1 = N_2 = N_3 = 15$ and $T_0 = 2.5$ sec.

0.0875, $\gamma_{\boldsymbol{\Omega}} = 0.11$ and $\zeta_{\boldsymbol{\Omega}} = 0.0219$, respectively. On the basis of these bounds, the estimated guaranteed stability margins of the GSHF based optimal regulator are quite tight and have the values

$$\mathrm{GM}^{\mathrm{in}}_{\mathrm{ext}} = (1 + \gamma_{\boldsymbol{\Omega}})^{-1} = 0.9009 \text{ or } -0.9067 \text{ dB},$$

$$\mathrm{GM}^{\mathrm{up}}_{\mathrm{ext}} = (1 - \gamma_{\boldsymbol{\Omega}})^{-1} = 1.1236 \text{ or } 1.0125 \text{ dB}$$

$$\mathrm{PM}_{\gamma} = \pm\arccos\left(1 - \frac{\gamma_{\boldsymbol{\Omega}}^2}{2}\right) = \pm 0.1101 \text{ rad or } \pm 6.3072^\circ$$

Consider now the case where the covariance kernel

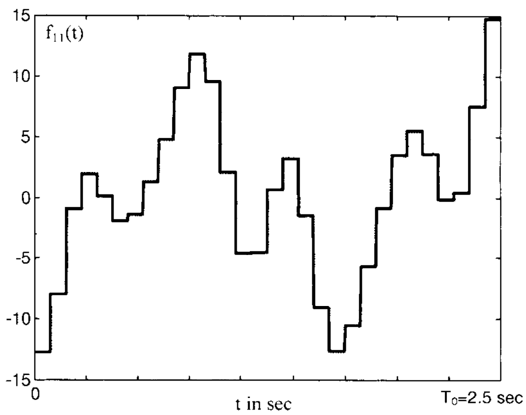

Figure 5. The 1-1 entry of $\mathbf{F}(t)$ in the case of a multirate GSHF with $N_1 = 30$, $N_2 = 45$, $N_3 = 60$ and $T_0 = 2.5$ *sec*.

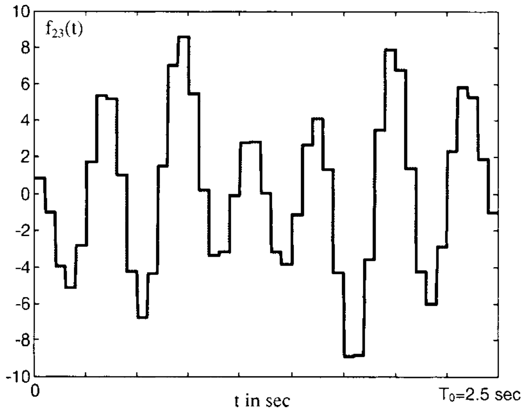

Figure 6. The 2-3 entry of $\mathbf{F}(t)$ in the case of a multirate GSHF with $N_1 = 30$, $N_2 = 45$, $N_3 = 60$ and $T_0 = 2.5$ sec.

$$\mathbf{R}_w = \begin{bmatrix} & \mathbf{0}_{3\times 6} \\ \mathbf{0}_{3\times 3} & 10^{-2} \times \mathbf{I}_{3\times 3} \end{bmatrix}$$

and all the other parameters remain the same as in the previous optimal design. In this case, guaranteed stability margins of the GSHF based optimal regulator can be obtained by applying Theorems 3.1 and 3.6, since the conditions and assumptions of Theorems 3.2-3.5 and 3.7 are not satisfied. Applying these results, we obtain the bounds $\alpha_{\boldsymbol{\Omega}} = 0.2676$ and $\zeta_{\boldsymbol{\Omega}} = 0.0692$, respectively. On the basis of these bounds, the estimated guaranteed stability margins of the above designed GSHF based optimal regulator are

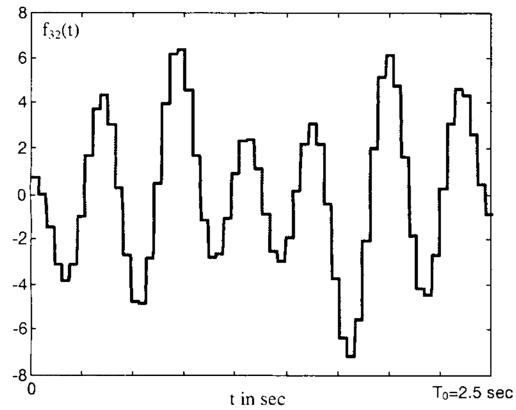

Figure 7. The 3-2 entry of $\mathbf{F}(t)$ in the case of a multirate GSHF with $N_1 = 30$, $N_2 = 45$, $N_3 = 60$ and $T_0 = 2.5$ sec.

$$\mathrm{GM}_{\mathrm{ext}}^{\mathrm{in}} = (1+\alpha_{\mathbf{\Omega}})^{-1} = 0.7889 \text{ or } -2.0598 \text{ dB},$$
$$\mathrm{GM}_{\mathrm{ext}}^{\mathrm{up}} = (1-\alpha_{\mathbf{\Omega}})^{-1} = 1.3654 \text{ or } 2.7053 \text{ dB}$$
$$\mathrm{PM}_{\mathrm{est}} = \pm\arccos(1-\frac{\alpha_{\mathbf{\Omega}}^2}{2}) = \pm 0.2684 \text{ rad or } \pm 15.3797°$$

Clearly, in the present case, the guaranteed stability margins of the GSHF based regulator are larger than the ones obtained in the previous case. In the case where the covariance kernel

$$\mathbf{R}_{\mathrm{w}} = \begin{bmatrix} & \mathbf{0}_{3\times 6} \\ \mathbf{0}_{3\times 3} & 10^{-3}\times \mathbf{I}_{3\times 3} \end{bmatrix},$$

the guaranteed stability margins of the regulator, obtained by applying Theorems 3.1 and 3.6 are

$$\mathrm{GM}_{\mathrm{ext}}^{\mathrm{in}} = (1+\alpha_{\mathbf{\Omega}})^{-1} = 0.6024 \text{ or } -4.4017 \text{ dB},$$
$$\mathrm{GM}_{\mathrm{ext}}^{\mathrm{up}} = (1-\alpha_{\mathbf{\Omega}})^{-1} = 2.9405 \text{ or } 9.3684 \text{ dB}$$
$$\mathrm{PM}_{\mathrm{est}} = \pm\arccos\left(1-\frac{\alpha_{\mathbf{\Omega}}^2}{2}\right) = \pm 0.6725 \text{ rad or } \pm 38.5327°.$$

In conclusion, when the covariance kernel $\mathbf{R}_{\mathrm{w}}$ decreases, the robustness of the GSHF based optimal regulator is ameliorated. Of course, this fact is expected, since a smaller covariance kernel $\mathbf{R}_{\mathrm{w}}$ corresponds to a smaller level of the noise disturbing the system. Then, the controller has to withstand to smaller system variations, and hence the closed-loop system becomes more robust. It is also worth noticing, at this point, that when the covariance kernel $\mathbf{R}_{\mathrm{w}}$ decreases, the regulator's gains also decrease in magnitude. This can be easily identified by Figs. 8 and 9, wherein the regulator's gains $f_{11}(t)$ and $f_{23}(t)$ are given for the case where

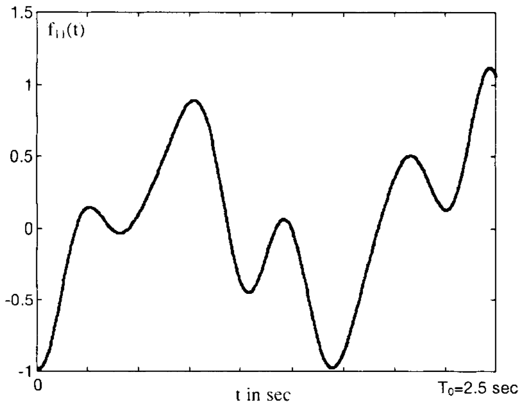

Figure 8. Hold function $f_{11}(t)$ for $T_0 = 2.5$ sec and $\mathbf{R}_{\mathrm{w}}$ decreased ($\mathbf{R}_{\mathrm{v}}$ increased) 10^3 times compared to (5.5).

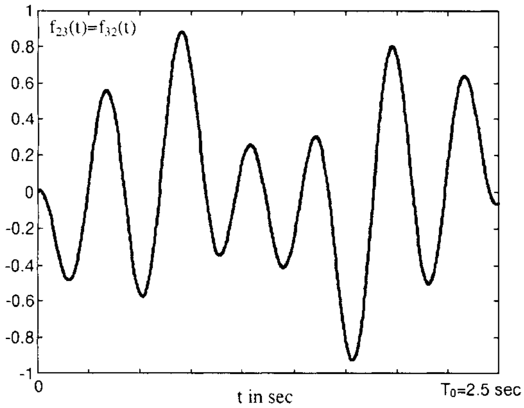

Figure 9. Gains $f_{23}(t)$, $f_{32}(t)$ for $T_0 = 2.5$ sec and $\mathbf{R}_w$ decreased ($\mathbf{R}_v$ increased) 10^3 times compared to (5.5).

$$\mathbf{R}_w = \begin{bmatrix} & \mathbf{0}_{3\times 6} \\ \mathbf{0}_{3\times 3} & 10^{-4} \times \mathbf{I}_{3\times 3} \end{bmatrix}.$$

Clearly, this is due to the fact that, the less the level of the disturbance acting on the system the smaller must be the control effort, for rejecting the disturbance.

We next consider the case where, the covariance kernel $\mathbf{R}_v = 2 \times \mathbf{I}_{3\times 3}$, and all the other parameters remain the same as in the original optimal design. Then, the regulator gains $f_{11}(t)$ and $f_{23}(t)$ are identical to those of Figs. 8 and 9. We can conclude that, if the covariance kernel $\mathbf{R}_v$ is large, small gains of the hold function $\mathbf{F}(t)$ are expected, indicating a poorly regulated system, since, the measurement noise predominates.

We finally study the impact of the sampling period T_0 on the gain $\mathbf{F}(t)$. To this end, in what follows, the sampling period is decreased to the value $T_0 = 0.5$ sec and the covariance kernels maintain the values given by (5.1). In this case, we obtain

$$\mathbf{K} = 10^{-4} \times \begin{bmatrix} 12 & 8 & 3 & 29 & 13 & 4 \\ & 8 & 4 & 13 & 19 & 9 \\ & & 5 & 4 & 9 & 15 \\ & \text{Symmetry} & & 385 & -6 & 25 \\ & & & & 416 & -31 \\ & & & & & 391 \end{bmatrix},$$

$$\tilde{\mathbf{F}}_\phi = \begin{bmatrix} -0.0877 & -0.0593 & -0.0510 \\ -0.0593 & -0.0794 & -0.0083 \\ -0.0510 & -0.0083 & -0.0284 \\ 0.1235 & 0.4023 & 0.7342 \\ 0.4023 & 0.4555 & -0.3320 \\ 0.7342 & -0.3320 & -0.2787 \end{bmatrix}.$$

The minimum of the cost function is $J_{\min} = tr\{\mathbf{KQ}\} = tr\{\mathbf{K}\} = 0.1218$. The

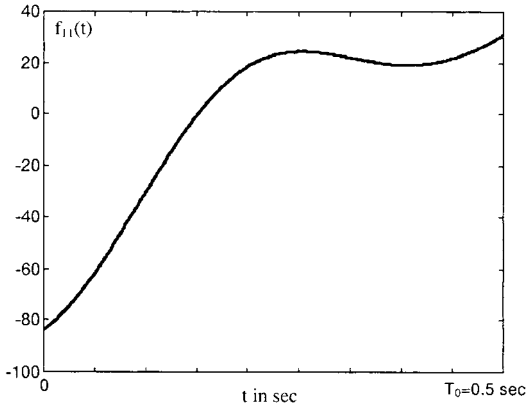

Figure 10. The 1-1 entry of $\mathbf{F}(t)$ in the case of $T_0 = 0.5$ sec.

elements $f_{11}(t)$ and $f_{23}(t)$ of the admissible hold function $\mathbf{F}(t)$ are depicted in Figs. 10 and 11, and they are larger than the gains obtained in the case where $T_0 = 2.5$ sec. This is due to the fact that, choosing a small T_0 yields a small $\mathbf{W}(\mathbf{A},\mathbf{B},\mathrm{T}_0)$ or a small $\hat{\mathbf{B}}$ (in the case of multirate GSHF). Their inversion produces matrices with large entries, which are involved in the computation of the gain $\mathbf{F}(t)$.

The results of Sections 2 and 3, together with the above observations, can be used to assess the detrimental effect of noise on the closed-loop system and the tradeoff involved in assuring good performance and sufficient robustness as well as in selecting the sampling period T_0.

Conclusions

In this paper, the GSHF based technique for optimal noise rejection has been

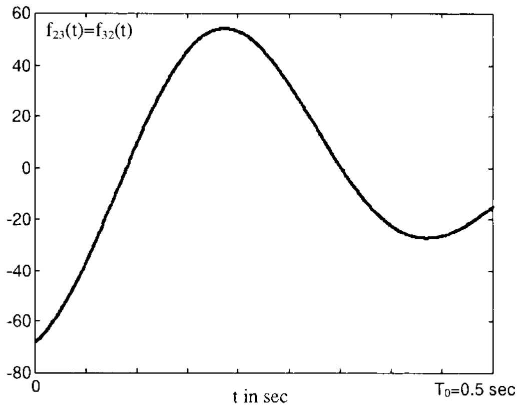

Figure 11. Entries 2-3 and 3-2 of $\mathbf{F}(t)$ in the case of $T_0 = 0.5$ sec.

applied to structural control and particularly to the IPB 800 three-story, single bay, steel frame structure. Moreover, an extensive analysis of the robustness properties of the GSHF based optimal regulator has been presented. The effectiveness of the method has been illustrated by various simulation results. From these results it has been recognized that when the covariance kernel of the disturbance acting on the structure is small, the regulator gains are small, since the less the level of the disturbance acting on the system the smaller must be the control effort, for rejecting the disturbance. Furthermore, large regulator gains are expected when the sampling period is chosen to be small. On the other hand, small regulator gains are expected in cases where the covariance kernel of the noise is large. This reveals that the covariance of the sampled state vector will be large, indicating a poorly regulated system in this case. It has also been concluded, through simulation, that when the covariance kernel of the noise disturbing the system decreases, the robustness of the GSHF based optimal regulator is ameliorated, since smaller disturbance covariance kernel correspond to smaller levels of system disturbances. In these cases, the controller withstands to smaller system variations, thus leading to a more robust closed-loop system. Overall, the theoretical and simulation results of the paper give some new insights to the GSHF control of linear systems and clearly indicate its effectiveness, when applied to the control of civil engineering structures.

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